Lecture 5, part 1: Reachability of secret distributions

Knowledge and Gossip — ESSLLI 2022

Malvin Gattinger (ILLC, Amsterdam)

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https://malv.in/2022/gossip/

Main reference

Hans van Ditmarsch, Malvin Gattinger, Ioannis Kokkinis, Louwe B. Kuijer:
Reachability of Five Gossip Protocols, RP 2019.
https://doi.org/10.1007/978-3-030-30806-3_17

Reachability

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Which **protocol** can reach which **distributions of secrets**?

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Motivation

- agents: which situations should I consider possible?
- outside observer: which protocol are they using?

Reachability: Definition

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$(ABCD, ABCD, ABC, ABD)$ is ANY-reachable, but not CO-reachable.

Why? 🤔

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W.l.o.g. the sequence must be $ab; ad; bc; ab$, but the last call ab is not CO-allowed!

Reminder: Protocols

		a may call *b* ...
LNS	Learn New Secrets	iff *a* does not have *b*'s secret.
CO	Call Once	once (either way).
TOK	Token	iff *a* has a token. Then *a* gives her token to *b*.
SPI	Spider	iff *a* has a token. Then *a* takes the token from *b*.
ANY	Any Call	at any time.

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ANY	Any Call	at any time.

Note: all of these are epistemic (agents know what calls they are allowed to make) and symmetric (no special roles, all agents are treated the same way).

Overview of Results

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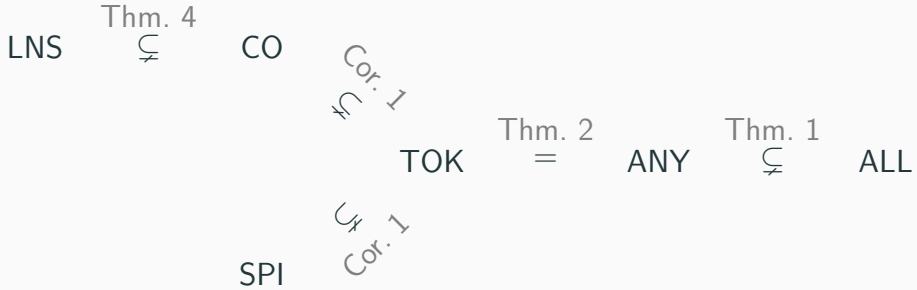

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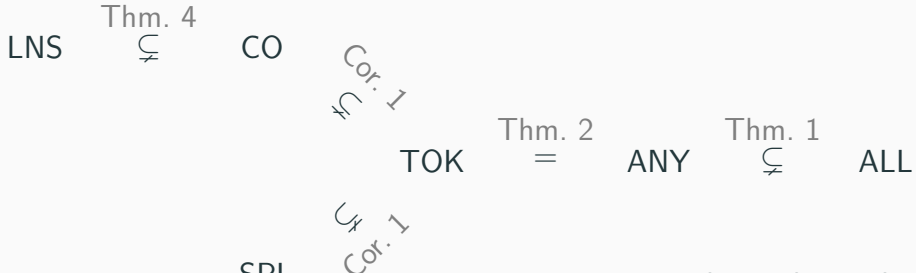

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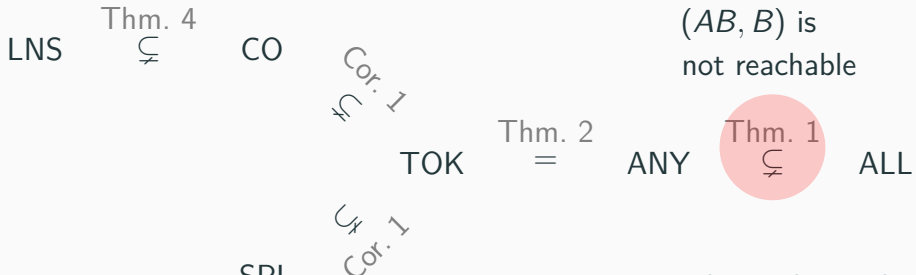

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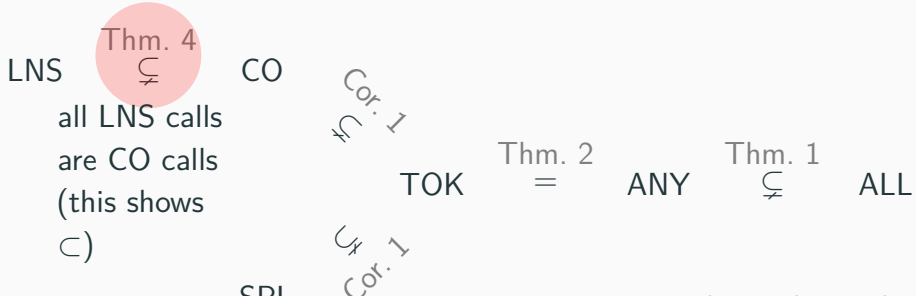

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LNS reaches a proper subset of CO

Theorem 4

There is a CO-reachable distribution that is not LNS-reachable, hence LNS $\subsetneq$ CO.

Proof.

$$t = (ABCDEF, ABC, ABCDE, ABCDEF, DEF, ABDEF) \ .$$

is wlog only reached by

$$ab; cb; ed; ef; cd; af; ad$$

but the last call is not LNS permitted!

TOK = ANY

Theorem 2

Every ANY-reachable distribution is TOK-reachable.

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Token Density Lemma

Let s be a TOK-reachable distribution and let a, b be two agents. Then s can be reached by a TOK-call sequence σ such that after the execution of σ at least one of a and b have a token.

SPI != CO

Theorem 3

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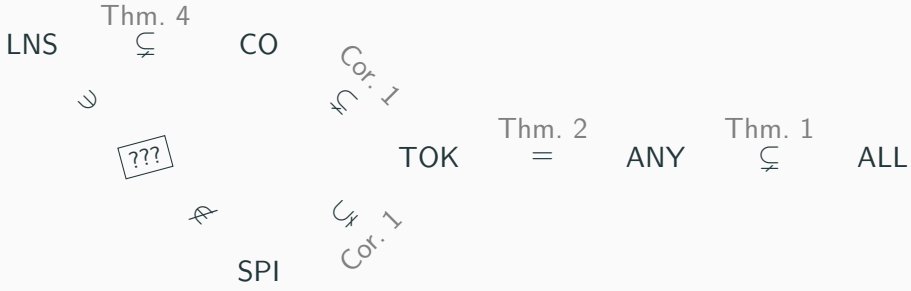
Proof. Consider $(ABCD, ABCD, ABC, ABD)$ and make case distinctions.

Distinguishing $LNS \subseteq CO \subseteq TOK$ from SPI



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Proof. Consider 16 agents $1, 2, \ldots, 8, a, b, \ldots, h$. Write (ab) for either ab or ba and consider

$$\begin{aligned}\sigma := \quad & (12); (34); (56); (78); (ab); (cd); (ef); (gh); \\ & (23); (45); (67); (81); (bc); (de); (fg); (ha); \\ & (1a); (4c); (7h); (6f)\end{aligned}$$

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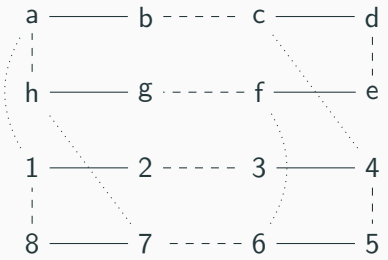
$$\begin{aligned}\sigma := \quad & (12);(34);(56);(78);(ab);(cd);(ef);(gh); && \text{solid lines}\\ & (23);(45);(67);(81);(bc);(de);(fg);(ha); && \text{dashed lines}\\ & (1a);(4c);(7h);(6f) && \text{dotted lines}\end{aligned}$$



Distinguishing $LNS \subseteq CO \subseteq TOK$ from SPI: example is not SPI

Proof. (continued)

First, check that σ is indeed LNS-permitted.

Suppose, towards a contradiction, that (some variant of) σ is SPI-permitted.

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1. Solid calls: callee of each pair loses its token.
2. Dashed calls: If 1 still has a token, then 2 does not. So 3 must have a token, otherwise (23) could not take place. Same for 4 and 5 etc.
   Hence, in both blocks, either all even or all odd agents lost their token.

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3. Dotted calls: $(1a)$, $(4c)$, $(7h)$ and $(6f)$ are all combinations of odd and even. Hence at least one of these calls is between two agents that both do not have a token any more. Hence that call is not SPI-permitted!

Distinguishing $LNS \subseteq CO \subseteq TOK$ from SPI: resulting distribution

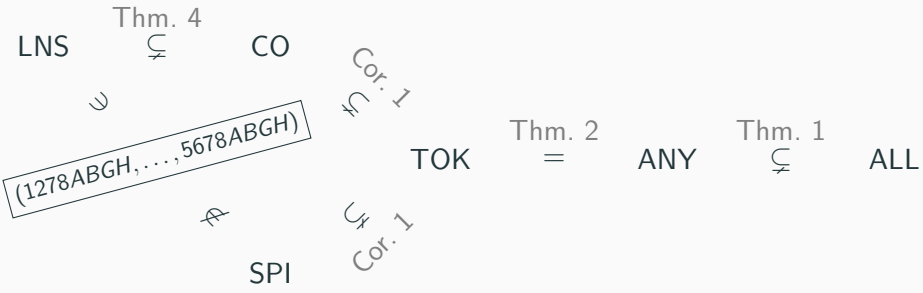
Finally, we can show that the resulting distribution is also not reachable by any other SPI-permitted sequence.

1 :	1278*ABGH*	5 :	3456	*a* :	1278*ABGH*	*e* :	*CDEF*
2 :	1234	6 :	5678*EFGH*	*b* :	*ABCD*	*f* :	5678*EFGH*
3 :	1234	7 :	5678*ABGH*	*c* :	3456*ABCD*	*g* :	*EFGH*
4 :	3456*ABCD*	8 :	1278	*d* :	*CDEF*	*h* :	5678*ABGH*

Claim

Except for the order of calls within the solid / dashed / dotted parts, and the call directions, only σ leads to the distribution above.

Distinguishing $LNS \subseteq CO \subseteq TOK$ from SPI using 16 agents!



Do we need so many agents?

Up to five agents, many protocols reach the same (number of non-isomorphic) distributions:

n	LNS	CO	SPI	TOK = ANY
2	2	2	2	2
3	4	4	4	4
4	15	15	16	16
5	97	97	111	111

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For LNS and ANY, see https://oeis.org/A307085 and https://oeis.org/A318154.
See (Ditmarsch, Kokkinis, and Stockmarr 2017) and (Kokkinis 2019) for the counter.

Subreachability

Subreachability — Why?

Motivation 1: Reasoning About Others

Agents might reason like this:

> *There are only two agents beside me and a call happened,*
> *so they now know each other's secrets.*

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Motivation 2: Logical Structure of Gossip

Given a syntax and semantics on gossip graphs, what are the validities?

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Given a syntax and semantics on gossip graphs, what are the validities?

$\Rightarrow$ This depends on our *class of models*.

Do we include models which are not reachable from initial graphs?

Subreachability

Definition

We *restrict* a distribution by forgetting/dropping some agents.

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Examples

- (AB, ABC, ABC) restricted to $\{A, C\}$ is (A, AC).
- $(ABC, ABCD, ACD, AD)$ restricted to $\{A, B, C\}$ is (ABC, ABC, AC).

Definition

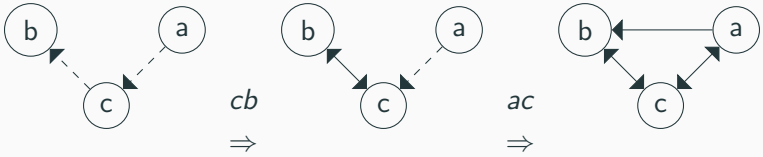
A distribution is P-subreachable if it is a restriction of a P-reachable distribution.

Some unreachable graphs are subreachable!

Example

This graph is not reachable from initial graphs.

But it is subreachable because we can start with the graph below and do $\sigma := (cb); (ac)$ to construct it.



Subreachability — Result

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Every finite gossip graph is subreachable.

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Definition

- For any $G = (A, N, S)$ let $\mathrm{Size}(G) := |A| + |N \setminus S| + |S \setminus \mathrm{id}_A|$.

  Intuitively, this is the number of things you draw.

Example

$\mathrm{Size}(G) = 3$

- We call $G = (A, N, S)$ *finite* iff $\mathrm{Size}(G) \in \mathbb{N}$.

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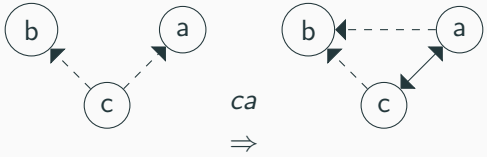

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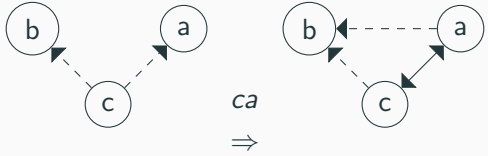

3. To add an $S \cap N$-edge from a to b, add someone who knows the number of b and whose number is known by a. Then first let c call b and at the end let a call c.

Formal Proof 1/2

Proof. By induction on $\mathrm{Size}(G)$.

Base case: $\mathrm{Size}(G) = 0$ for $G = (A, N, S)$. Then $A = N = S = \varnothing$. Witness: $G_0 := (\varnothing, \varnothing, \varnothing)$ and $\sigma := \epsilon$.

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Induction step: Take G' such that $\text{Size}(G') = k + 1$.

Let G be a subgraph of G' such that either

1. G' has one disconnected agent more than G,
2. G' has one $N \setminus S$ edge more than G, or
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In all cases $\mathrm{Size}(G) = k$, so by induction hypothesis G is subreachable.
Hence there are $G_0 = (A_0, N_0, \mathrm{id}_A)$ and σ such that $G \sqsubseteq (G_0)^\sigma$.

Formal Proof 2/2

Now, we consider the three cases:

1. If G' has one disconnected agent more than G, say c, then let $G'_0 := (A_0 \cup \{c\}, N_0, \mathrm{id}_{A_0 \cup \{c\}})$ and $\sigma' := \sigma$.

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In each case we can check that $G' \sqsubseteq (G'_0)^{\sigma'}$. Hence G' is subreachable. □

Subreachability with protocols.

Theorem 6

All graphs are P-subreachable for any of the five P above.

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Proof. The previous construction can be tweaked to yield P-permitted sequences.

Conclusion

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Summary

- TOK and ANY reach the same distributions, all others differ, with some inclusions.
- Given enough agents, all distributions are *sub*reachable by any protocol.

Additional References

Ditmarsch, H. van, I. Kokkinis, and A. Stockmarr. 2017. "Reachability and Expectation in Gossiping." In *Proceedings of the 20th PRIMA*, edited by B. An, A. Bazzan, J. Leite, S. Villata, and L. van der Torre, 93–109. Springer. https://doi.org/10.1007/978-3-319-69131-2/_6.

Kokkinis, Ioannis. 2019. "Implementation for Reachability and Expectation in Gossiping." https://github.com/Jannis17/gossip_protocol_expectation.