Lecture 4: Strengthening

Knowledge and Gossip — ESSLLI 2022

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2022-08-11, Galway

https://malv.in/2022/gossip/

Main reference:

Hans van Ditmarsch, Malvin Gattinger, Louwe B. Kuijer, Pere Pardo. 2019. "Strengthening Gossip Protocols using Protocol-Dependent Knowledge" *Journal of Applied Logics* 6 (1): 157-203. https://malv.in/2019/StrengtheningGossipProtocols.pdf

Motivation

When does LNS work?

Theorem (yesterday / BIMS paper)

LNS is

- strongly successful iff G is a "sun graph"
- weakly successful iff G is not a "bush" or a "double bush"

Can we improve LNS? – Strengthening Protocols

Depending on the graph, LNS can be strongly or weakly successful!

Can we make it better?

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Informal Idea

Only make a call iff LNS allows it

and you know that it leads to a good situation.

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Can we make it better?

Informal Idea

Only make a call iff LNS allows it

and you know that it leads to a good situation.

But how do you *know* which calls are okay?

Reminder: Protocol-dependent knowledge

Reminder: Syntax

The *language* of protocol-dependent knowledge:

$$\varphi ::= \top \mid N_i i \mid S_i i \mid C_i i \mid i = i \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i^P \varphi \mid [\pi]\varphi$$

$$\pi ::= ?\varphi \mid ii \mid \pi ; \pi \mid \pi \cup \pi \mid \pi^*$$

Definition

A *protocol* is a function P mapping any agent pair ab to a formula P_{ab} called the *protocol condition*.

Example

The *Learn New Secrets* (LNS) protocol is $LNS_{ab} := \neg S_a b$.

Reminder: Semantics

A *state* is a tuple (G, σ) where $G = (A, N, S)$ is an initial graph and σ a call sequence.

Let N^σ and S^σ be the resulting relations after executing σ.

$$
\begin{array}{lcl}
G, \sigma \models N_x y & :\Leftrightarrow & (x, y) \in N^\sigma \\
G, \sigma \models S_x y & :\Leftrightarrow & (x, y) \in S^\sigma \\
G, \sigma \models C_x y & :\Leftrightarrow & xy \in \sigma \text{ or } yx \in \sigma \\
G, \sigma \models x = y & :\Leftrightarrow & x = y \\
G, \sigma \models K_a^P \varphi & \text{iff} & G, \sigma' \models \varphi \text{ for all } (G, \sigma') \sim_a^P (G, \sigma) \\
G, \sigma \models [\pi]\varphi & \text{iff} & G, \sigma' \models \varphi \text{ for all } (G, \sigma') \in [\![\pi]\!](G, \sigma)
\end{array}
$$

$$
\begin{array}{rcl}
[\![?\varphi]\!](G, \sigma) & := & \{(G, \sigma) \mid G, \sigma \models \varphi\} \\
[\![ab]\!](G, \sigma) & := & \{(G, (\sigma; ab)) \mid G, \sigma \models N_a b\} \\
[\![\pi; \pi']\!](G, \sigma) & := & \bigcup\{[\![\pi']\!](G, \sigma') \mid (G, \sigma') \in [\![\pi]\!](G, \sigma)\} \\
[\![\pi \cup \pi']\!](G, \sigma) & := & [\![\pi]\!](G, \sigma) \cup [\![\pi']\!](G, \sigma) \\
[\![\pi^*]\!](G, \sigma) & := & \bigcup\{[\![\pi^n]\!](G, \sigma) \mid n \in \mathbb{N}\}
\end{array}
$$

Reminder: Epistemic Alternatives

The easy definition, without protocols:

Definition

For any agent a and protocol P let $\sim_a$ be the smallest relation such that:

- $(G, \epsilon) \sim_a (G, \epsilon)$;
- if $(G, \sigma) \sim_a (G, \tau)$, $N_b^\sigma = N_b^\tau$, $S_b^\sigma = S_b^\tau$,
  then $(G, \sigma; ab) \sim_a (G, \tau; ab)$;
  if $(G, \sigma) \sim_a (G, \tau)$, $N_b^\sigma = N_b^\tau$, $S_b^\sigma = S_b^\tau$,
  then $(G, \sigma; ba) \sim_a (G, \tau; ba)$;
- if $(G, \sigma) \sim_a (G, \tau)$ and $a \notin \{c, d, e, f\}$,
  then $(G, \sigma; cd) \sim_a (G, \tau; ef)$.

Note: We only do *synchronous* here.

Reminder: Protocol-dependent Epistemic Alternatives

The tricky definition, with protocols

Definition

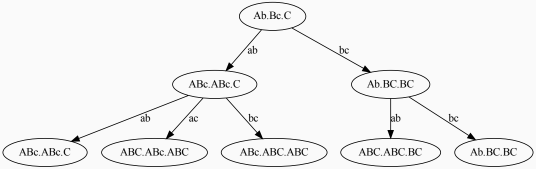
For any agent a and protocol P let $\sim_a^P$ be the smallest relation such that:

- $(G, \epsilon) \sim_a^P (G, \epsilon)$;
- if $(G, \sigma) \sim_a^P (G, \tau)$, $N_b^\sigma = N_b^\tau$, $S_b^\sigma = S_b^\tau$, and $G, \sigma \models P_{ab}$ and $G, \tau \models P_{ab}$,
  then $(G, \sigma; ab) \sim_a^P (G, \tau; ab)$;
  if $(G, \sigma) \sim_a^P (G, \tau)$, $N_b^\sigma = N_b^\tau$, $S_b^\sigma = S_b^\tau$, and $G, \sigma \models P_{ba}$ and at $G, \tau \models P_{ab}$,
  then $(G, \sigma; ba) \sim_a^P (G, \tau; ba)$;
- if $(G, \sigma) \sim_a^P (G, \tau)$ and $a \notin \{c, d, e, f\}$ such that $G, \sigma \models P_{cd}$ and $G, \tau \models P_{ef}$,
  then $(G, \sigma; cd) \sim_a^P (G, \tau; ef)$.

Note: We only do *synchronous* here.

Common knowledge of a protocol prunes the execution tree!

```
GoMoChe> pdf $ treeUpTo 2 (wlog anyCall) (lineInit 3, [])
```



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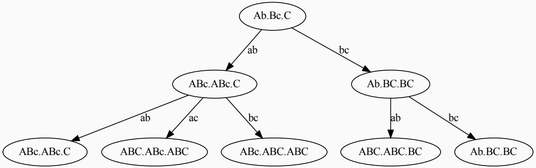
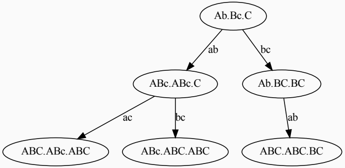



```
GoMoChe> pdf $ treeUpTo 2 (wlog lns) (lineInit 3, [])
```



What can our Language express?

After call ab, they know each others secret: $[ab](S_a b \wedge S_b a)$

Everyone knows all secrets: $Ex := \bigwedge_{i,j} S_i j$

After any of three calls, everyone knows all secrets: $[ab \cup bc \cup ac]Ex$

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Learn-New-Secrets condition: $\text{LNS}_{ab} := \neg S_a b$

LNS protocol:

$$\text{LNS} := \left(\bigcup_{a \neq b \in A} (?(N_a b \wedge \neg S_a b); ab) \right)^* ; ? \bigwedge_{a \neq b \in A} \neg (N_a b \wedge \neg S_a b)$$

LNS is strongly successful: $[\text{LNS}]Ex$

LNS is weakly successful: $\langle \text{LNS} \rangle Ex$

Improving LNS with Epistemic Logic

$\mathrm{LNS}_{ab} := \neg S_a b$

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Idea: Make call if LNS allows it, and you know that it leads to a good situation.

What is a good situation?

- LNS can still succeed: $\langle LNS \rangle Ex$

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We define the **hard strengthening** of *LNS* by:

$$\text{LNS}^{\blacksquare}_{ab} := LNS_{ab} \wedge K_a^{\text{LNS}} [ab] \langle LNS \rangle Ex$$

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Historic side note: This is actually why we found/invented K^P in the first place, to avoid self-reference.

If you still worry about Russell here, see Lecture 2 and main reference.

Four different Syntactic Strengthenings

Given protocol: P_{ab}

Hard

$$P^{\blacksquare}_{ab} := P_{ab} \wedge K^P_a[ab]\langle P\rangle Ex$$

Soft

$$P^{\blacklozenge}_{ab} := P_{ab} \wedge \hat{K}^P_a[ab]\langle P\rangle Ex$$

Hard Step-wise

$$P^{\square}_{ab} := P_{ab} \wedge K^P_a[ab](Ex \vee \bigvee_{i,j}(N_ij \wedge P_{ij}))$$

Soft Step-wise

$$P^{\lozenge}_{ab} := P_{ab} \wedge \hat{K}^P_a[ab](Ex \vee \bigvee_{i,j}(N_ij \wedge P_{ij}))$$

Semantic Strengthening: Uniform Backward Induction

Instead of syntactically defining a strengthening, we can also work semantically on the tree or set of call sequences directly!

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One semantic strengthening is from Game and Decision Theory:

Definition: Uniform Backward Induction/Defoliation ("hard" version)

For any set of sequences X, let $\mathrm{UBI}_P(X)$ be X without $\sigma; ab$ such that

- there is a $\sigma' \in X$ such that
  - $(G, \sigma') \sim_a^P (G, \sigma)$ and
  - $\sigma'; ab$ is terminal in X and
  - $(G, \sigma'; ab) \not\models Ex$.

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  - $(G, \sigma'; ab) \not\models Ex$.

This is also known as "common knowledge of stable belief in rationality" (Baltag, Smets, and Zvesper 2009) or "common belief in future rationality" (Perea 2014).

$\square = \text{UBI}$

Theorem

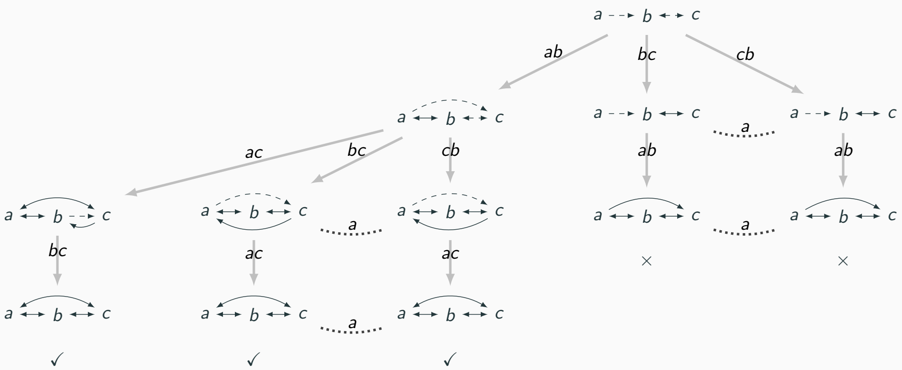
Step-wise Strengthening is the same as Uniform Backward Induction:

$$P^{\square}(G) = \text{UBI}_P(P(G))$$

Good News

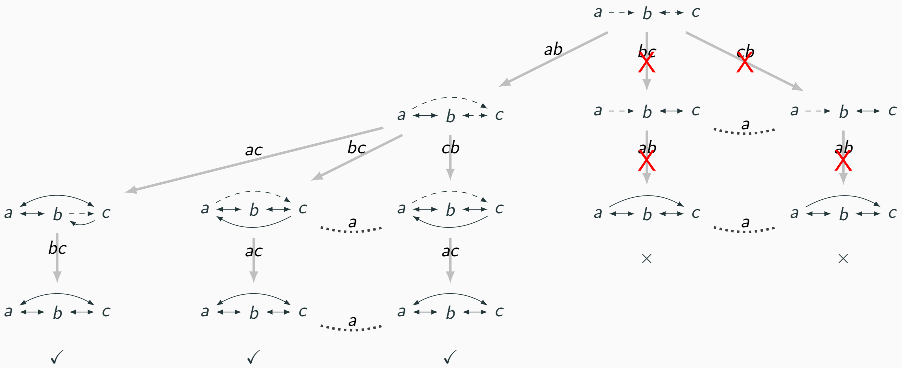
Example of Strongly Successful Strengthening

With LNS:



Example of Strongly Successful Strengthening II

With hard strengthening of LNS:



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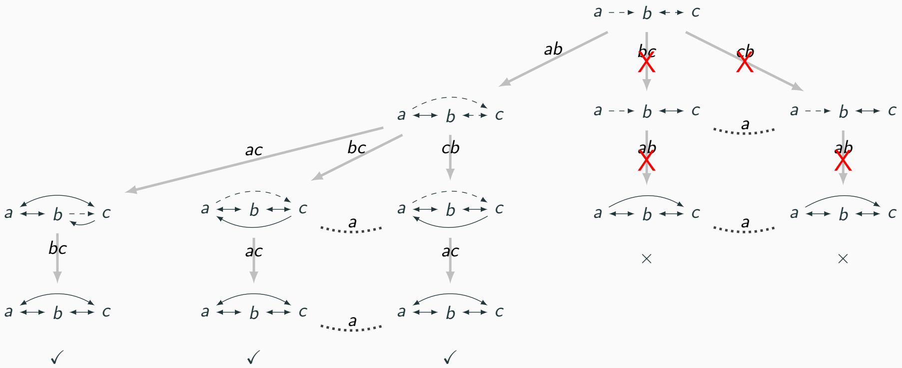
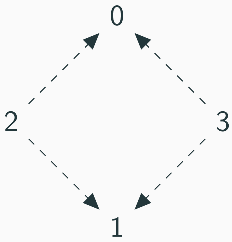


Note: The strengthening "repairs" LNS for this example, but not in general!

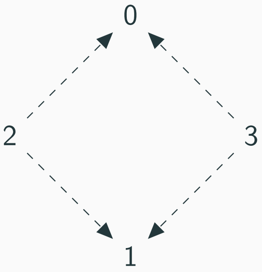
The Diamond Example I



All LNS-sequences up to the decision point:

20; 01	×	21; 10	×	30; 01	×	31; 10	×
20; 21	×	21; 20	×	30; 20; 01	✓	31; 20	✓
20; 30; 01	✓	21; 30	✓	30; 20; 21	✓	31; 21; 10	✓
20; 30; 21	×	21; 31; 10	✓	30; 20; 31	×	31; 21; 20	✓
20; 30; 31	✓	21; 31; 20	×	30; 21	✓	31; 21; 30	×
20; 31	✓	21; 31; 30	✓	30; 31	×	31; 30	×

The Diamond Example II



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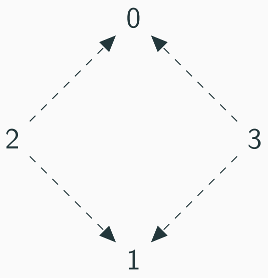
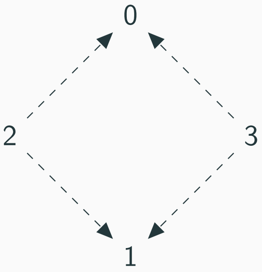


Protocol	successful	unsuccessful
LNS	48	44
$LNS^{\blacksquare}$	8	8
$LNS^{\blacksquare 2}$	0	4
$LNS^{\blacksquare 3}$	0	0
$LNS^{\blacklozenge}$	48	8
$LNS^{\blacklozenge 2}$	48	8
$LNS^{\blacklozenge 3}$	48	8
$LNS^{\square}$	24	36
$LNS^{\square 2}$	8	16
$LNS^{\square 3}$	8	4
$LNS^{\square 4}$	0	4
$LNS^{\square 5}$	0	0
$LNS^{\lozenge}$	48	36
$LNS^{\lozenge 2}$	48	32
$LNS^{\lozenge 3}$	48	32

The Diamond Example II



But there is another LNS strengthening which is strongly successful here!
$\Rightarrow$ Exercise!

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Bad News

The Question

Is there a *perfect strengthening* of LNS?

Formally, is there a protocol which strengthens LNS to become strongly successful on all graphs where the original LNS is weakly successful?

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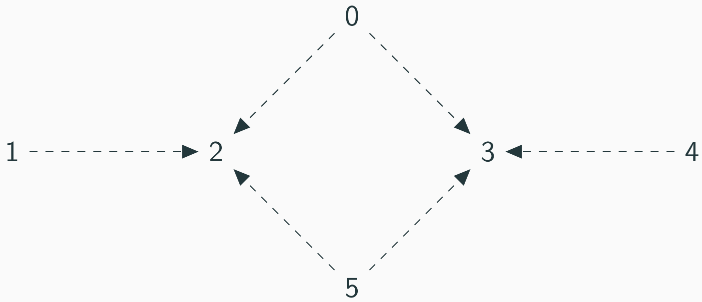
Hint: No.

The Diamond with Hands aka Candy



Claim

LNS is weakly successful on this graph, but there is no epistemic symmetric protocol that is a strengthening of LNS and that is strongly successful on this graph.

Proof by Exhaustive Search

- LNS is weakly successful here:
  - 02; 12; 53; 43; 13; 03; 23; 52; 42 is successful
  - 02; 12; 53; 43; 13; 03; 52; 42 is unsuccessful

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- How to check all of them?

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- How to check all of them? Using GoMoChe, obviously ;-)

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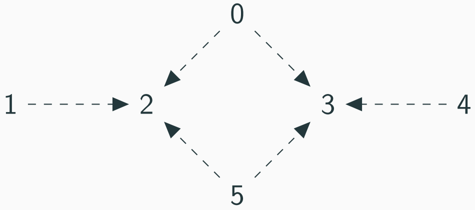
  - 02; 12; 53; 43; 13; 03; 52; 42 is unsuccessful
- There are 9468 LNS-sequences for the given graph.
- How to check all of them? Using GoMoChe, obviously ;-)

We use a combination of model checking and “manual” proof by case distinction . . .

Proof Idea

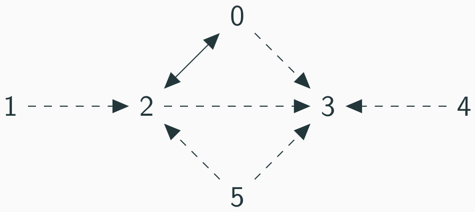
Suppose there is a perfect strengthening of LNS on this graph.



What could be a successful sequence of calls allowed by that protocol?

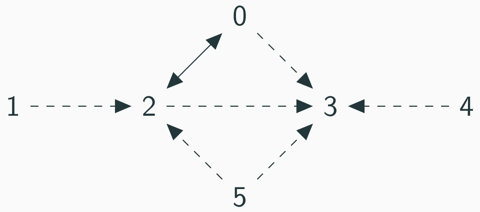
- 0, 1, 4 and 5 do not have incoming arrows $\Rightarrow$ they will never be called.
- If 1 calls 2 first, then 1 never becomes an expert, same for 4 and 3.
- Hence, w.l.o.g. the first call is 02

Proof Idea II



- After 02, can we continue with 12?
- First call could have been 03 which *looks the same* to agent 1.
- But 03; 12 is losing, since then 1 cannot become an expert

Proof Idea II



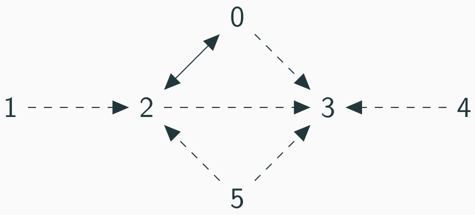
- After 02, can we continue with 12?
- First call could have been 03 which *looks the same* to agent 1.
- But 03; 12 is losing, since then 1 cannot become an expert

(We now use that the protocol is symmetric and epistemic.)

- If 03; 12 is not allowed, also 02; 12 must be forbidden.

⇒ we cannot continue with 12

Proof Idea III



More formally, suppose our new protocol condition is P_{ab}:

$(G, 02) \sim_1 (G, 03)$ implies that $(G, 02) \models P_{12}$ iff $(G, 03) \models P_{12}$

But we must have $(G, 03) \not\models P_{12}$.

Now continue with a lot more case distinctions like this ...

An Impossibility Result

Theorem

There is *no* epistemic protocol which strengthens LNS to become strongly successful on all graphs where the original LNS is weakly successful.

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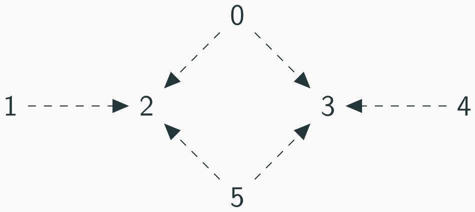
There is *no* epistemic protocol which strengthens LNS to become strongly successful on all graphs where the original LNS is weakly successful.

Note about generality: This theorem is *not* syntax/language dependent. It applies to all (semantic) strengthenings of LNS, even those not in our language.

So what happens if we do it anyway?



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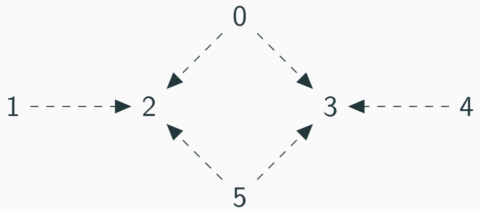

If we apply hard strengthening to this graph, only the first 02 call is allowed. Afterwards we have an empty protocol.

Strengthening in GoMoChe

Strengthening in GoMoChe I

```
GoMoChe> lns
(\(z,y) -> Neg (S z y))
```

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```
GoMoChe> lns
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GoMoChe> strengHard lns
(\(v,u) -> Conj
  [ Neg (S v u)
  , K v (\(z,y) -> Neg (S z y))
    (Box (Call v u)
         (Dia
           (Seq [ Star (CupAg (\y -> CupAg (\z -> Cup
                    [ Seq [ Test (Neg (Same y z))
                          , Seq [Test (Conj [N y z,Neg (S y z)]),Call y z] ]
                    , Seq [ Test (Neg (Neg (Same y z)))
                          , Test Bot ] ])))
                , Test (ForallAg (\y -> ForallAg (\z -> Disj
                    [ Same y z
                    , Disj [Neg (N y z),Neg (Neg (S y z)) ] ]))) ] )
           (ForallAg (\y -> ForallAg (\z -> S y z))) ) ] )
```

Strengthening in GoMoChe II

```
type Strengthening = Protocol -> Protocol

strengHard, strengSoft, strengStepHard, strengStepSoft :: Strengthening
strengHard     p (a,b) = Conj [p (a,b) , K    a p $ Box (Call a b) (Dia (protoTerm p) allExperts)]
strengSoft     p (a,b) = Conj [p (a,b) , HatK a p $ Box (Call a b) (Dia (protoTerm p) allExperts)]
strengStepHard p (a,b) = Conj [p (a,b) , K    a p $ Box (Call a b) (Disj [allExperts, protoCanGoOn p])]
strengStepSoft p (a,b) = Conj [p (a,b) , HatK a p $ Box (Call a b) (Disj [allExperts, protoCanGoOn p])]
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Another strengthening, relevant for tomorrow:

```haskell
super :: Protocol -> Protocol
super proto (x, y) = Conj [ Neg (superExpert x cmo) , proto (x,y) ]
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See `src/Gossip/Strengthening.hs`,

Strengthening in GoMoChe II

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strengHard, strengSoft, strengStepHard, strengStepSoft :: Strengthening
strengHard     p (a,b) = Conj [p (a,b) , K    a p $ Box (Call a b) (Dia (protoTerm p) allExperts)]
strengSoft     p (a,b) = Conj [p (a,b) , HatK a p $ Box (Call a b) (Dia (protoTerm p) allExperts)]
strengStepHard p (a,b) = Conj [p (a,b) , K    a p $ Box (Call a b) (Disj [allExperts, protoCanGoOn p])]
strengStepSoft p (a,b) = Conj [p (a,b) , HatK a p $ Box (Call a b) (Disj [allExperts, protoCanGoOn p])]
```
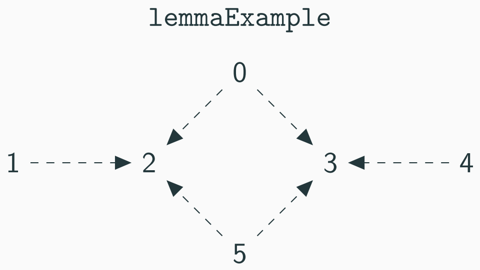

Another strengthening, relevant for tomorrow:

```
super :: Protocol -> Protocol
super proto (x, y) = Conj [ Neg (superExpert x cmo) , proto (x,y) ]
```

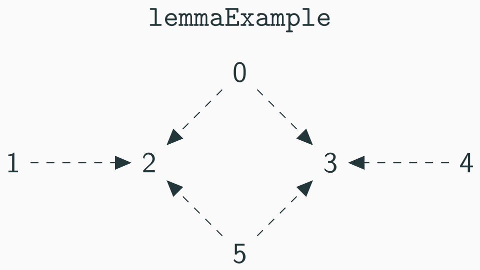
See `src/Gossip/Strengthening.hs`, in particular `diamondProto` and `diamondProtoOld` for a protocol that *is* strongly successful on the diamond example 😺
(and thus not a strengthening of LNS).

Diamond with Hands in GoMoChe




```
GoMoChe> isWeaklySucc localLns lemmaExample
True
```

Diamond with Hands in GoMoChe




```
GoMoChe> isWeaklySucc localLns lemmaExample
True
```

Hard strengthening of LNS is empty after 02 (this takes a while to compute!):

```
GoMoChe> tree (strengHard lns) (lemmaExample,[(0,2)]) == Node (lemmaExample,[(0,2)]) []
True
```

Diamond with Hands in GoMoChe II

```
GoMoChe> showTreeUpToDecision (tree lns (lemmaExample, []))
023-12-2-3-34-235 I6
     (0,2): 023-12-023-3-34-235 02-1-02-3-4-5
          (0,3): ϟ 186
          (1,2): 023-0123-0123-3-34-235 02-012-012-3-4-5
               (0,3): ϟ 76
               (1,3): ϟ 76
               (2,3): ϟ 48
               (4,3): ϟ 96
               (5,2): ϟ 120
               (5,3): 023-0123-0123-235-34-235 02-012-012-35-4-35
                    (0,3): ϟ 18
                    (1,3): ϟ 18
                    (2,3): ϟ 24
                    (3,2): ϟ 24
                    (4,3): 023-0123-0123-2345-2345-235 02-012-012-345-345-35
                         (0,3): ϟ 14
                         (1,3): 023-012345-0123-012345-2345-235 02-012345-012-012345-345-35
                              (0,3): 012345-012345-0123-012345-2345-235 012345-012345-012-012345-345-35
                                   (2,3): ☺ 2
                                   (4,2): ☺ 1
                                   (5,2): ϟ 1
                              (2,3): ☺ 6
                              (4,2): ☺ 2
                              ...
```

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Open Questions

- *How* good are step-wise strengthenings? (They are easier to compute!)
- Is there an incomparable but "LNS-like" protocol that beats LNS on many/most/all graphs?
- Is there a complete axiomatization for K^P (in general, not just for gossip!)?
- When are self-referential strengthenings well-defined?

  $P^*_{ab} := P_{ab} \wedge K_a^{P^*}[ab]\langle P \rangle Ex$

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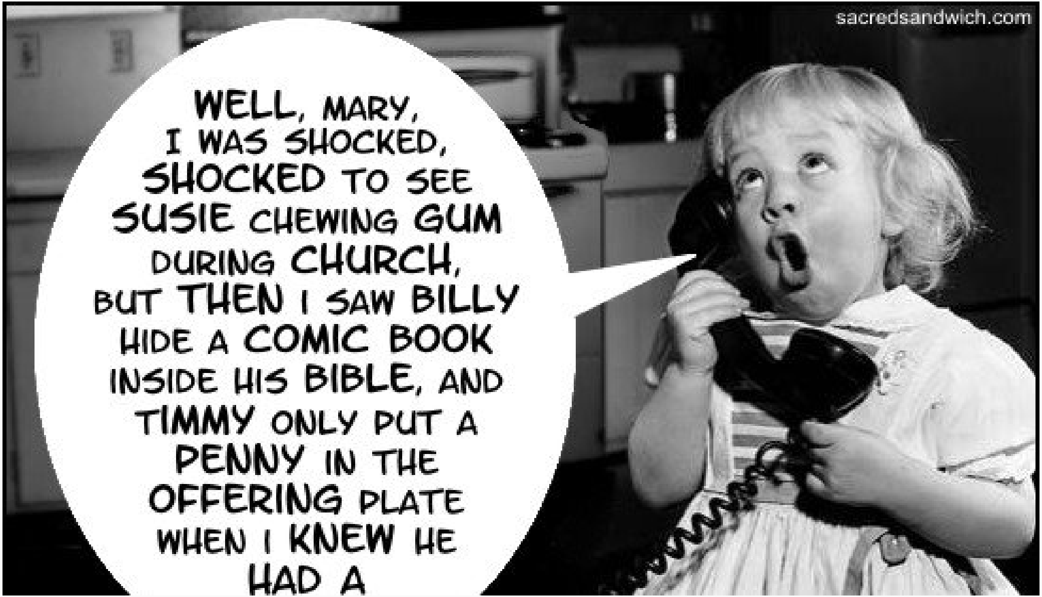
sacredsandwich.com
WELL, MARY, I WAS SHOCKED, SHOCKED TO SEE SUSIE CHEWING GUM DURING CHURCH, BUT THEN I SAW BILLY HIDE A COMIC BOOK INSIDE HIS BIBLE, AND TIMMY ONLY PUT A PENNY IN THE OFFERING PLATE WHEN I KNEW HE HAD A

References

Main reference:

Hans van Ditmarsch, Malvin Gattinger, Louwe B. Kuijer, Pere Pardo. 2019. "Strengthening Gossip Protocols using Protocol-Dependent Knowledge" *Journal of Applied Logics* 6 (1): 157-203. https://malv.in/2019/StrengtheningGossipProtocols.pdf

Other references:

Baltag, Alexandru, Sonja Smets, and Jonathan Alexander Zvesper. 2009. "Keep 'Hoping' for Rationality: A Solution to the Backward Induction Paradox." *Synthese* 169 (2): 301–33. https://doi.org/10.1007/s11229-009-9559-z.

Perea, Andrés. 2014. "Belief in the Opponents' Future Rationality." *Games and Economic Behavior* 83 (Supplement C): 231–54. https://doi.org/10.1016/j.geb.2013.11.008.