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A Note on Constructible Dynamic Gossip Graphs

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Abstract

The classical gossip problem studies how long it takes for n agents to spread n secrets. It is assumed that all agents only know their own secret in the beginning but everyone is able to call everyone. Dynamic gossip generalizes this by restricting the possible calls according to a graph that says who has the phone number of whom. In a call the agents exchange both secrets and numbers, thereby changing the graph.

Given these rules of dynamic gossip, some situations or graphs are unreachable. For example, if we only consider two agents Alice and Bob, then it it can not happen that Alice knows the secret of Bob but not vice versa. However, the situation changes if we consider configurations of subgraphs. Among three agents the asymmetric situation can occur, depending on calls involving the third one.

This raises the question which graphs can occur as subgraphs in a situation with more agents. In particular this is relevant if we model situations in which the number of agents is unknown or their reasoning power limited. In this talk we prove a simple constructive answer: All finite gossip graphs can be constructed by starting with an appropriate initial graph and making specific calls.

Based on other work and discussions with

- Hans van Ditmarsch,
- Jan van Eijck,
- Louwe B. Kuijer,
- Pere Pardo and
- Rahim Ramezanian.

Contents

1. What is Dynamic Gossip?

The Gossip Problem

Suppose n agents each have a secret.

In a phone call two agents exchange all secrets they know.

- How many calls until everyone knows all secrets?

Well-known: $2n - 4$ calls are necessary and sufficient.

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- How many calls until everyone knows all secrets?

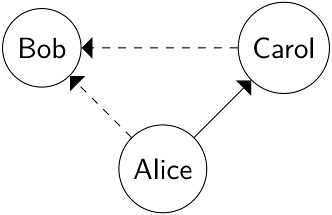
Well-known: $2n - 4$ calls are necessary and sufficient.

Crucial assumption: Everyone can call everyone else.

Dynamic Gossip: Graphs

Not everyone can call everyone.

- A *gossip graph* is a triple $G = (A, N, S)$ where
  - A is the set of agents,
  - $N \subseteq A \times A$ is the number relation (dashed),
  - $S \subseteq A \times A$ is the secret relation (solid), and
  - we have $\mathrm{id}_A \subseteq S \subseteq N$.

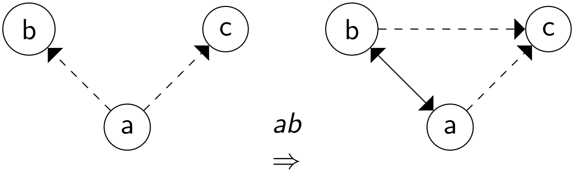


- A gossip graph $G = (A, N, S)$ is called *initial* iff $S = \mathrm{id}_A$.

Dynamic Gossip: Calls

Secrets and phone numbers are exchanged.

- A *call* is an ordered pair xy. It is *possible* in $G = (A, N, S)$ iff Nxy.
- If xy is possible in $G = (A, N, S)$, then let $G^{xy} := (A, N', S')$ where
  - $N' := N \cup \{(x, y), (y, x)\} \circ N$ and
  - $S' := S \cup \{(x, y), (y, x)\} \circ S$.



Call Sequences and Reachability

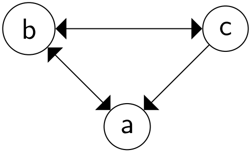
- A *calling sequence* is a list of calls $\sigma \in (A \times A)^*$. We write ϵ for the empty sequence.
- The *result* of σ on G is given by $G^\epsilon := G$ and $G^{\sigma;xy} := (G^\sigma)^{xy}$. (Only if xy is possible on G^σ, otherwise it is undefined.)
- $G = (A, N, S)$ is *reachable from an initial graph* iff there are $N_0 \subseteq N$ and a compatible σ such that $(A, N_0, \mathrm{id})^\sigma = G$.

Question: Which graphs are reachable?

2. What can happen?

Call Sequences and Reachability: Example

Can this situation happen?



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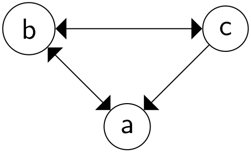
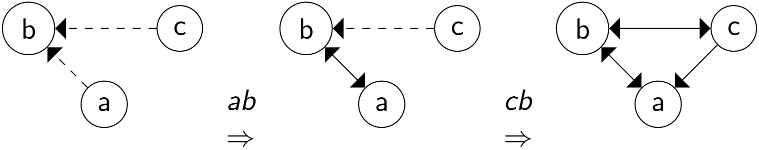


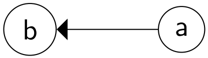
I.e. is it reachable from an initial graph? Yes: $G = {G_0}^{ab;cb}$



Not Everything Can Happen

Example

This graph is not reachable from an initial graph with two agents:



There are examples for any number of agents.

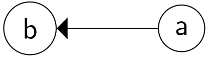
Subgraphs

- $G = (A, N, S)$ is a *subgraph* of $G' = (A', N', S')$, short $G \sqsubseteq G'$ iff
  - $A \subseteq A'$,
  - for all $a, b \in A$ we have Nab iff $N'ab$, and Sab iff $S'ab$.
- A gossip graph G is *constructible as a subgraph*, short *caas* iff there is an initial gossip graph $G_0 = (A_0, N_0, S_0)$ and a calling sequence σ over A_0 such that $G \sqsubseteq (G_0)^\sigma$.

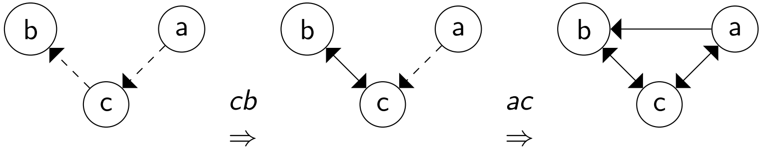
Some Unreachable Graphs Are Constructible

Example

This graph is not reachable from initial graphs.



But it is *caas* because we can start with the graph below and do $\sigma := (cb);(ac)$ to construct it.



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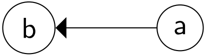

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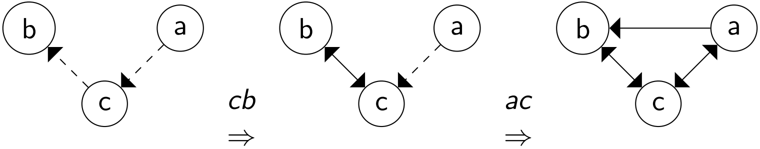

Question: Which things are constructible?

3. Why do we care?

Motivation 1: Reasoning About Others

Consider the dynamic gossip scenario where the number of agents is not known to some of the agents or their reasoning power is limited so they can not think about all agents at the same time.

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Consider the dynamic gossip scenario where the number of agents is not known to some of the agents or their reasoning power is limited so they can not think about all agents at the same time.

$\Rightarrow$ Then agents can no longer use reasoning steps like

There are only two other agents and a call happened, so now they must have each others secret.

Motivation 2: Logical Structure of Gossip

Given a syntax and semantics on gossip graphs, what are the validities?

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Given a syntax and semantics on gossip graphs, what are the validities?

$\Rightarrow$ This depends on our *class of models*.

Do we include models which are not reachable from initial graphs?

4. What is caas?

Everything is caas!

Claim

Every finite gossip graph is caas.

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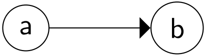
Definition

- For any $G = (A, N, S)$ let $\text{Size}(G) := |A| + |N \setminus S| + |S \setminus \text{id}_A|$. Intuitively, this is the number of things you draw.

Example

$\text{Size}(G) = 3$



- We call $G = (A, N, S)$ *finite* iff $\text{Size}(G) \in \mathbb{N}$.

Proof Idea

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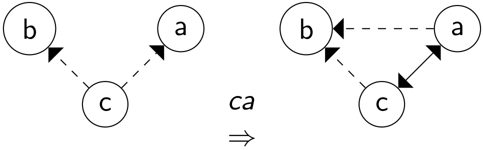

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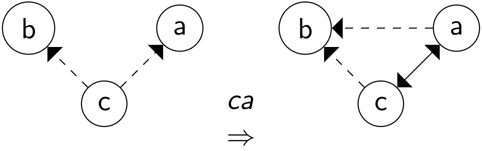

3. To add an $S \cap N$-edge from a to b, add someone who knows the number of b and whose number is known by a. Then first let c call b and at the end let a call c.

Formal Proof 1/2

Proof. By induction on Size(G).

Base case: Size(G) $= 0$ for $G = (A, N, S)$. Then $A = N = S = \varnothing$.
Witness: $G_0 := (\varnothing, \varnothing, \varnothing)$ and $\sigma := \epsilon$.

Induction hypothesis: Suppose all G with Size(G) $= k$ are caas.

Induction step: Take G' such that Size(G') $= k + 1$.

Let G be a subgraph of G' such that either

1. G' has one disconnected agent more than G,
2. G' has one $N \setminus S$ edge more than G, or
3. G' has one $S \cap N$ edge more than G.

In all cases Size(G) $= k$, so by induction hypothesis G is caas.

Formal Proof 2/2

Hence there are $G_0 = (A_0, N_0, \mathrm{id}_A)$ and σ such that $G \sqsubseteq (G_0)^\sigma$.

Now, we consider the three cases:

1. If G' has one disconnected agent more than G, say c, then let $G'_0 := (A_0 \cup \{c\}, N_0, \mathrm{id}_{A_0 \cup \{c\}})$ and $\sigma' := \sigma$.
2. If G' has one $N \setminus S$ edge more than G, say $(a, b) \in (N \setminus S)$, let c be a fresh agent,
   $G'_0 := (A_0 \cup \{c\}, N_0 \cup \{(c, a), (c, b)\}, \mathrm{id}_{A_0 \cup \{c\}})$ and
   $\sigma' := \sigma; (ca)$.
3. If G' has one $S \cap N$ edge more than G, say $(a, b) \in (N \cap S)$, let c be a fresh agent,
   $G'_0 := (A_0 \cup \{c\}, N_0 \cup \{(a, c), (c, b)\}, \mathrm{id}_{A_0 \cup \{c\}})$ and
   $\sigma' := (cb); \sigma; (ac)$.

In each case we can check that $G' \sqsubseteq (G'_0)^{\sigma'}$. Hence G' is caas. $\square$

Conclusion

Theorem
Every finite gossip graph is caas.

Informal Corollary
Suppse a formal gossip logic can not "count" agents.
Then it is sound and complete for the class of all graphs iff it is sound and complete for the class of reachable graphs.

Future Work:

- Improve Construction / Show Minimality
- Gossip Logic

Thank You!