MuBDDy Children.

Symbolic Model Checking DEL.

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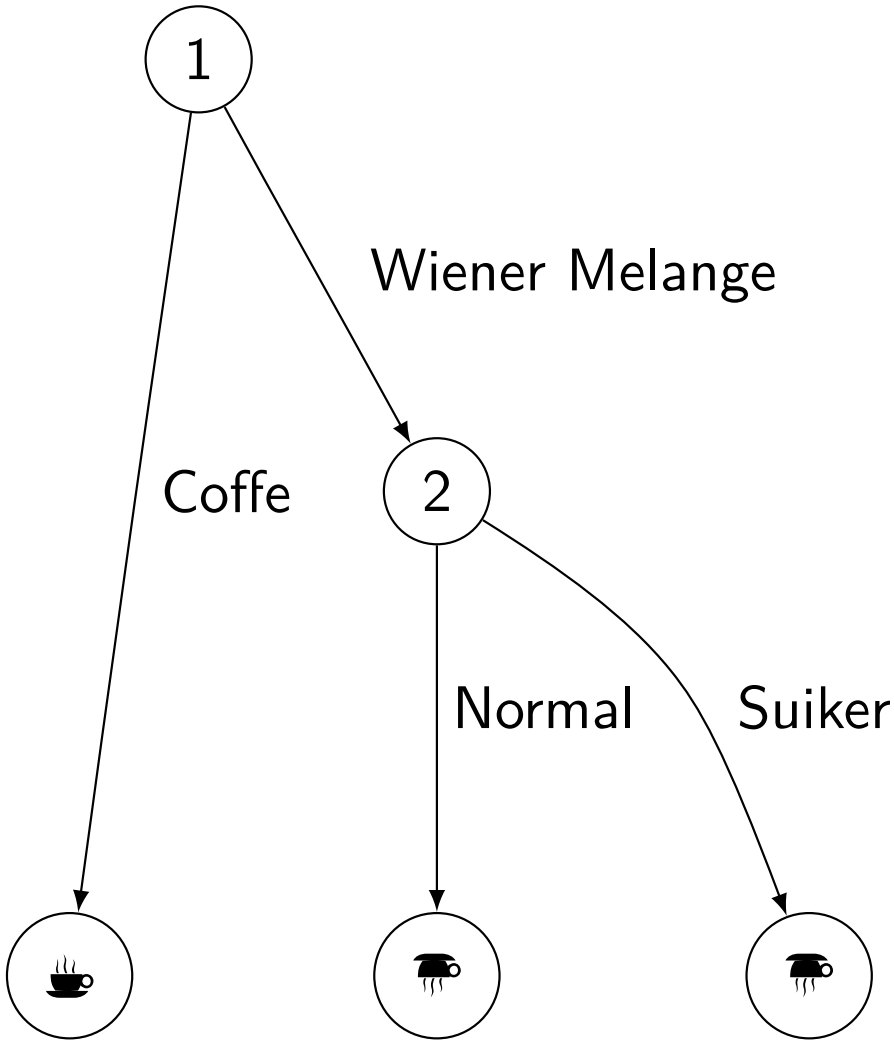
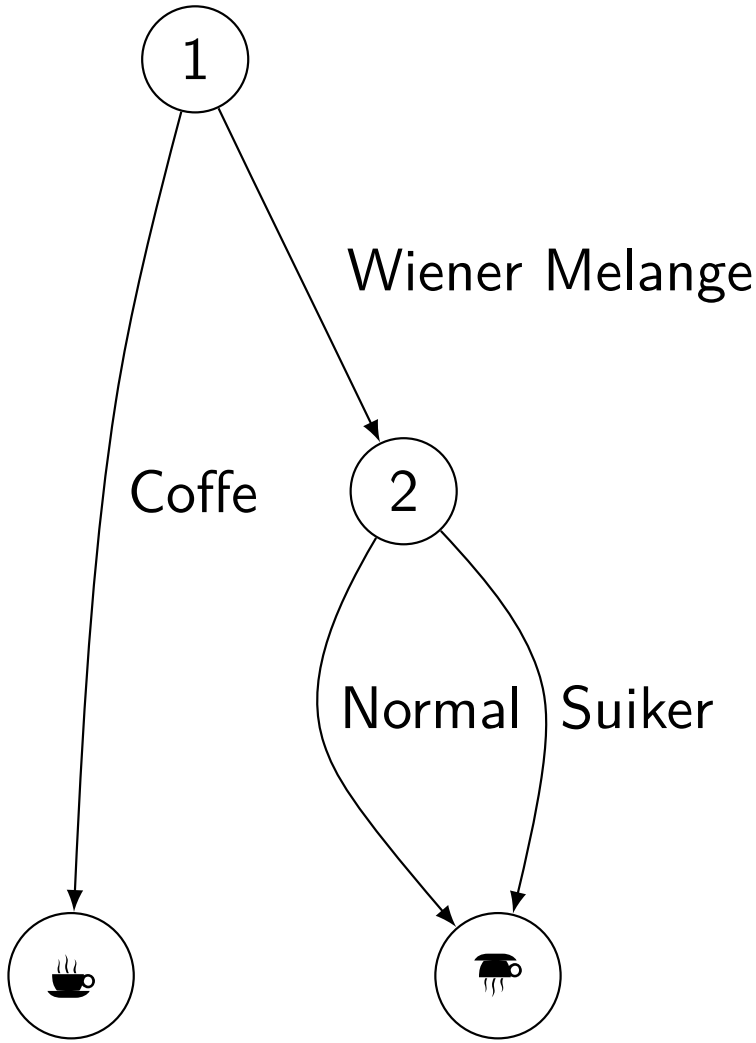
Outline

Coffee Machines

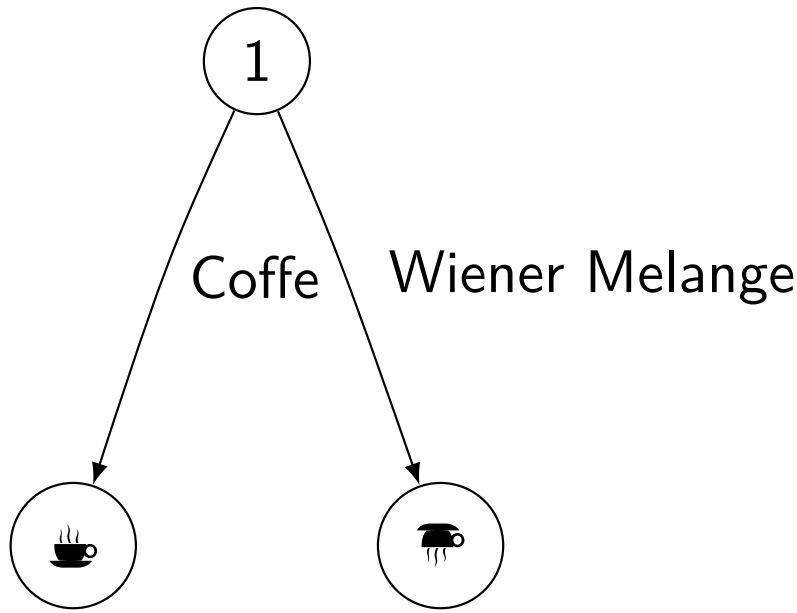
Coffee machines are annoying.



Coffee machines are still annoying.



Coffee machines do not have to be annoying.



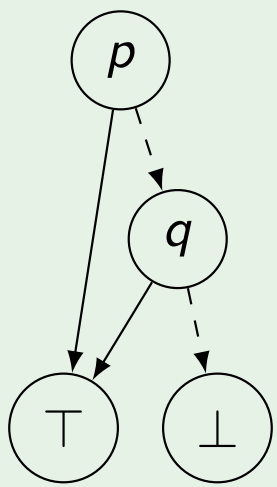
Binary Decision Diagrams

Binary Decision Diagrams

Definition

A Binary Decision Diagram [Bry86] for the variables V is a directed acyclic graph where non-terminal nodes are from V with two outgoing edges and terminal nodes are $\top$ or $\bot$.
Reduced: No redundancy, identify isomorphic subgraphs.
Ordered: Variables in a given order, maximally once.
By "BDD" we always mean a reduced and ordered BDD.

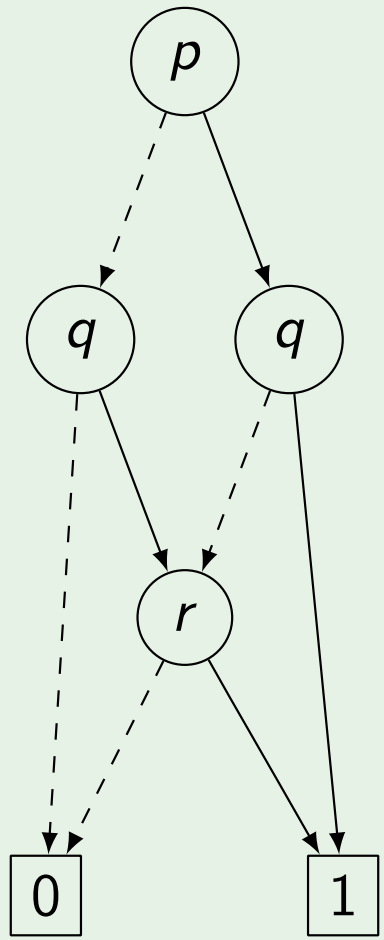
Example



Binary Decision Diagrams

Example

$$(p \wedge q) \vee (q \wedge r) \vee (p \wedge r)$$



Binary Decision Diagrams: Fun facts

Reduced and ordered BDDs provide a compact representation of boolean functions. Nice properties:

- Small: A BDD often takes less memory than a truth table.
- Canonical: All *equivalent* formulas have *identical* BDDs.
- Composable: Given the BDDs for ϕ and ψ we can easily find those for $\neg\phi$, $\phi \wedge \psi$, $\phi \vee \psi$, $\forall p\phi$, ...
- Simultaneous composition: A BDD for $\forall p(\phi \to \psi)$ can be cosntructed from BDDs for ϕ and ψ "in one go".

But: Only boolean! How to use BDDs for dynamic logics?
We generalize ideas in [SSL07] and [LSSC08] to cover full DEL.

Knowledge Structures

Interlude: Our DEL for today.

Definition

The DEL language without common knowledge is:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid K_i\phi \mid [!\phi]\phi \mid [\phi]_\Delta\phi$$

where $p \in V$, $i \in I$ and $\Delta \subseteq I$. Abbreviations for $\vee$, $\rightarrow$ and $\leftrightarrow$.
The *boolean formulas* over V are:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi$$

Fact

Using Kripke Models $\mathcal{M} = (W, (R_i)_{i\in I}, \mathrm{Val})$ *we can give semantics to this language and inductively define* $\mathcal{M}, w \vDash \phi$.

Knowledge Structures

Kripke models: Which situations can you distinguish?
Knowledge structure: What facts do you observe?

Definition

A knowledge structure is a tuple $\mathcal{F} = (V, \theta, O_1, \cdots, O_n)$ where

- *Vocabulary*: V is a set of propositional variables
- *Law*: θ is a boolean formula over V
- *Observational variables*: for each agent i, $O_i \subseteq V$

An assignment $s \subseteq V$ is a *state* of $\mathcal{F}$ iff it satisfies θ.
A pair $(\mathcal{F}, s)$ is called a scenario.
The local state for i at s is $s \cap O_i$.

Example

$\mathcal{F} = (V = \{p\}, \theta = \top, O_1 = \{p\}, O_2 = \varnothing)$

DEL Semantics on Knowledge Structures

Definition

Fix $\mathcal{F} = (V, \theta, O_1, \ldots, O_n)$ and a state s of $\mathcal{F}$.

$$
\begin{array}{lcl}
(\mathcal{F}, s) \models p & \iff & p \in s \\
(\mathcal{F}, s) \models \neg\alpha & \iff & \text{not } (\mathcal{F}, s) \models \alpha \\
(\mathcal{F}, s) \models \alpha \wedge \beta & \iff & (\mathcal{F}, s) \models \alpha \text{ and } (\mathcal{F}, s) \models \beta \\
(\mathcal{F}, s) \models K_i\alpha & \iff & \text{for all states } s' \text{ of } \mathcal{F}: \\
 & & \text{if } s \cap O_i = s' \cap O_i, \text{ then } (\mathcal{F}, s') \models \alpha \\
(\mathcal{F}, s) \models [!\psi]\alpha & \iff & (\mathcal{F}, s) \models \psi \text{ implies } (\mathcal{F}^{!\psi}, s) \models \alpha \\
 & & \text{where } \|\psi\|_{\mathcal{F}} \text{ is on the next slide} \\
 & & \text{and } \mathcal{F}^{!\psi} = (V, \theta \wedge \|\psi\|_{\mathcal{F}}, O_1, \cdots, O_n) \\
(\mathcal{F}, s) \models [\psi]_\Delta\alpha & \iff & (\mathcal{F}, s) \models \psi \text{ implies } (\mathcal{F}^\Delta_\psi, s \cup p_\psi) \models \alpha \\
 & & \text{where } p_\psi \text{ is fresh, } \|\psi\|_{\mathcal{F}} \text{ on next slide,} \\
 & & \mathcal{F}^\Delta_\psi := (V \cup p_\psi, \theta \wedge (p_\psi \to \|\psi\|_{\mathcal{F}}), O'_i) \\
 & & \text{and } O'_i = O_i \cup \{p_\psi | i \in \Delta\}
\end{array}
$$

Everything is Boolean!

Definition

Fix $\mathcal{F} = (V, \theta, O_1, \cdots, O_n)$.
Translate $\alpha \in \mathcal{L}(V)$ to a boolean formula $\|\alpha\|_{\mathcal{F}}$:

$$\begin{array}{lcl} p & \mapsto & p \\ \neg\beta & \mapsto & \neg\|\beta\|_{\mathcal{F}} \\ \beta_1 \wedge \beta_2 & \mapsto & \|\beta_1\|_{\mathcal{F}} \wedge \|\beta_2\|_{\mathcal{F}} \\ K_i\beta & \mapsto & \forall(V \setminus O_i)(\theta \to \|\beta\|_{\mathcal{F}}) \\ [!\psi]\beta & \mapsto & \|\psi\|_{\mathcal{F}} \to \|\beta\|_{\mathcal{F}^{!\psi}} \\ [\psi]_\Delta\beta & \mapsto & \|\psi\|_{\mathcal{F}} \to (\|\beta\|_{\mathcal{F}^{\psi}_{\Delta}})(\frac{p_\psi}{\mathbf{true}}) \end{array}$$

Lemma

For all scenarios $(\mathcal{F}, s)$ and all formulas ϕ:

$$(\mathcal{F}, s) \models \phi \iff (\mathcal{F}, s) \models \|\phi\|_{\mathcal{F}}$$

This means we can use BDDs for all KNS and formulas ☺

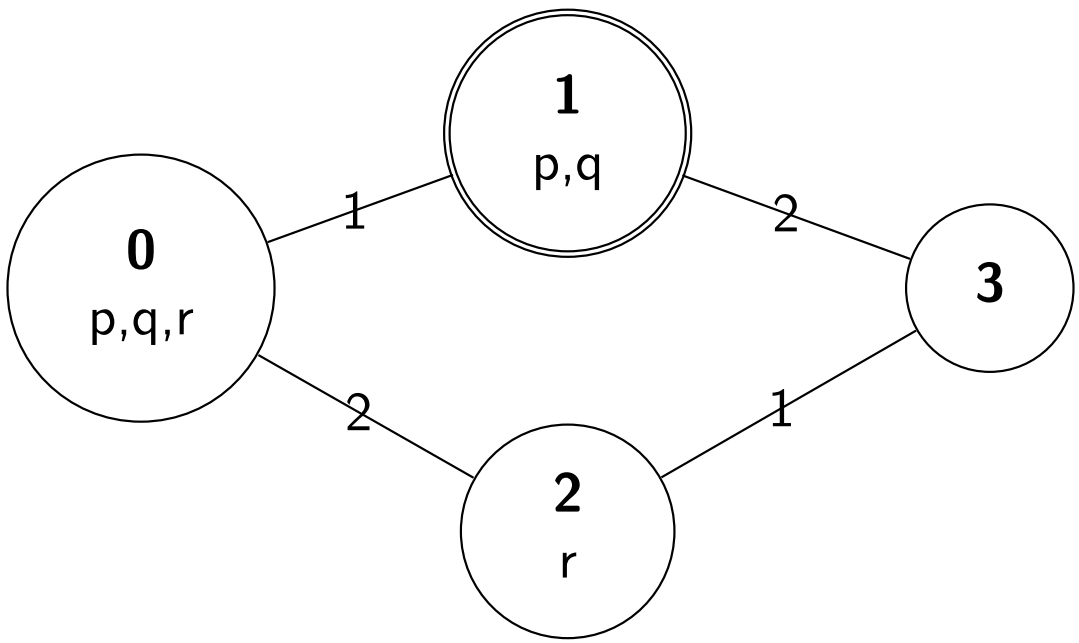
Knowledge Structures are S5 Kripke Models

Example

Consider this knowledge structure:

$$\mathcal{F} = (V = \{p, q, r\}, \theta = p \leftrightarrow q, O_1 = \{q\}, O_2 = \{r\})$$

Here we have $\mathcal{F}, \{p, q\} \models K_1 q$ and $\mathcal{F}, \{p, q\} \models \neg K_2 q$.



Knowledge Structures are S5 Kripke Models

Theorem

For any knowledge structure $\mathcal{F} = (V, \theta, O_i)$ there is an equivalent S5 Kripke model, namely:

$$\mathcal{M} = (W := \mathcal{P}(V), R_i := \{(s,t) \mid s \cap O_o = t \cap O_i\}, Val := \mathrm{id})$$

Theorem

For any S5 Kripke model, there is an equivalent knowledge structure. (Idea: Label all equivalence classes with (sets of) additional propositions to describe the observations.)

Muddy Children

Muddy Children

Father: “At least one of you has mud on their face.”
Father: “Do you know if you have mud on your face?”
Nobody reacts.
Father: “Do you know if you have mud on your face?”
Nobody reacts.
Father: “Do you know if you have mud on your face?”
All three children: “Yes, I do have.”

Muddy Children in DEL

The fact that after "Do you know if you have mud on your face?" nobody reacts means that nobody knows their own state:

$$\psi := \bigwedge_{i \in I} (\neg (K_i p_i \vee K_i \neg p_i))$$

The whole story:

$$[!(p_1 \vee p_2 \vee p_3)][!\psi][!\psi](\bigwedge_{i \in I} (K_i p_i)$$

For details and the standard analysis with Kripke semantics, see for example [DHK07, p. 93-96].

Muddy Children on Knowledge Structures

Initial Model:

$$\mathcal{F}_0 = \left(\{p_1, p_2, p_3\}, \top, \begin{array}{l} O_1 = \{p_2, p_3\} \\ O_2 = \{p_1, p_3\} \\ O_3 = \{p_1, p_2\} \end{array} \right)$$

Announcing a formula means to add ($\wedge$) it to the law (θ) of the knowledge structure. After "At least one of you is dirty":

$$\mathcal{F}_1 = \left(\{p_1, p_2, p_3\}, p_1 \vee p_2 \vee p_3, \begin{array}{l} O_1 = \{p_2, p_3\} \\ O_2 = \{p_1, p_3\} \\ O_3 = \{p_1, p_2\} \end{array} \right)$$

Muddy Children on Knowledge Structures

After "At least one of you is dirty.":

$$\mathcal{F}_1 = \left(\{p_1, p_2, p_3\}, p_1 \vee p_2 \vee p_3, \begin{array}{l} O_1 = \{p_2, p_3\} \\ O_2 = \{p_1, p_3\} \\ O_3 = \{p_1, p_2\} \end{array} \right)$$

Announcing an epistemic formula means to add its boolean equivalent to the law of the knowledge structure.
After "Nobody knows their own state.":

$$\mathcal{F}_2 = \left(\{p_1, p_2, p_3\}, (p_1 \vee p_2 \vee p_3) \wedge \|\psi\|_{\mathcal{F}_1}, \begin{array}{l} O_1 = \{p_2, p_3\} \\ O_2 = \{p_1, p_3\} \\ O_3 = \{p_1, p_2\} \end{array} \right)$$

Muddy Children on Knowledge Structures

$$
\begin{aligned}
\|K_1 p_1\|_{\mathcal{F}_1} &= \forall(V \setminus O_1)(\theta_1 \to \|p_1\|_{\mathcal{F}_1}) \\
&= \forall p_1((p_1 \vee p_2 \vee p_3) \to p_1) \\
&= ((\top \vee p_2 \vee p_3) \to \top) \wedge ((\bot \vee p_2 \vee p_3) \to \bot) \\
&= \top \wedge (\neg(p_2 \vee p_3)) \\
&= \neg(p_2 \vee p_3)
\end{aligned}
$$

$$
\begin{aligned}
\|K_1 \neg p_1\|_{\mathcal{F}_1} &= \forall(V \setminus O_1)(\theta_1 \to \|\neg p_1\|_{\mathcal{F}_1}) \\
&= \forall p_1((p_1 \vee p_2 \vee p_3) \to \neg p_1) \\
&= ((\top \vee p_2 \vee p_3) \to \neg\top) \wedge ((\bot \vee p_2 \vee p_3) \to \neg\bot) \\
&= (\top \to \bot) \wedge ((p_2 \vee p_3) \to \top) \\
&= \bot
\end{aligned}
$$

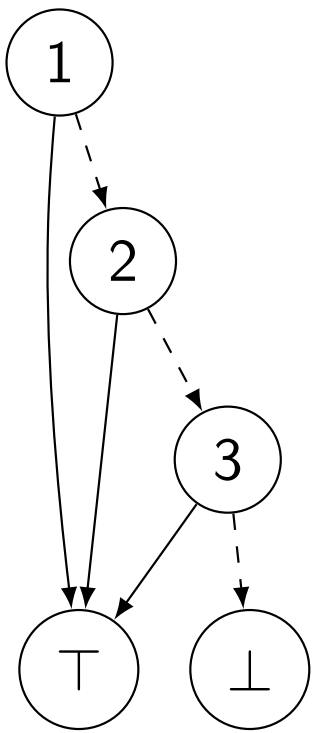
Similarly for agents 2 and 3. The announced formula then becomes

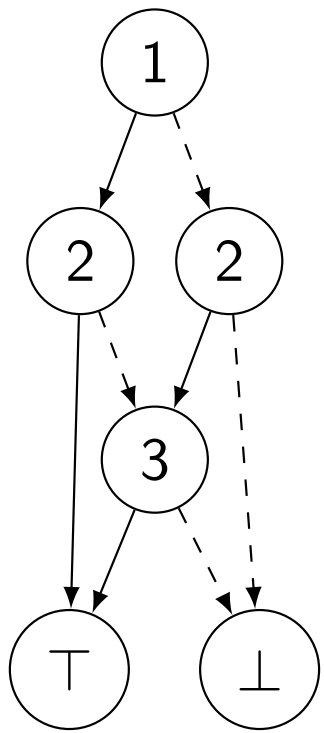
$$
\begin{aligned}
\|\psi\|_{\mathcal{F}_1} &= \|\bigwedge_{i \in I}(\neg(K_i p_i \vee K_i \neg p_i))\|_{\mathcal{F}_1} = \bigwedge_{i \in I} \|\neg(K_i p_i \vee K_i \neg p_i)\|_{\mathcal{F}_1} \\
&= \neg(\neg(p_2 \vee p_3)) \wedge \neg(\neg(p_1 \vee p_3)) \wedge \neg(\neg(p_1 \vee p_2)) \\
&= (p_2 \vee p_3) \wedge (p_1 \vee p_3) \wedge (p_1 \vee p_2)
\end{aligned}
$$

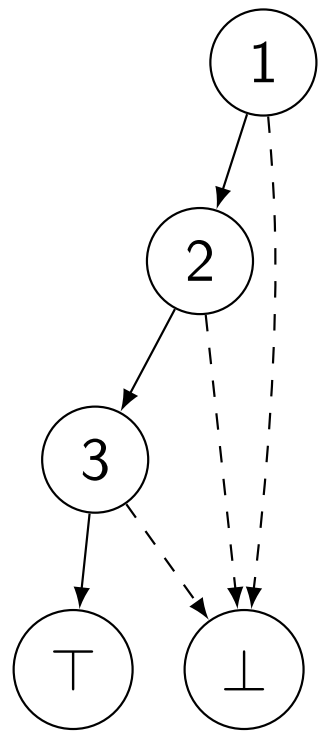
Muddy Children on Knowledge Structures

$$\mathcal{F}_k = \left(\{p_1, p_2, p_3\}, \theta_k, \begin{array}{l} O_1 = \{p_2, p_3\} \\ O_2 = \{p_1, p_3\} \\ O_3 = \{p_1, p_2\} \end{array} \right)$$

where θ_1, θ_2 and θ_3 are given by:







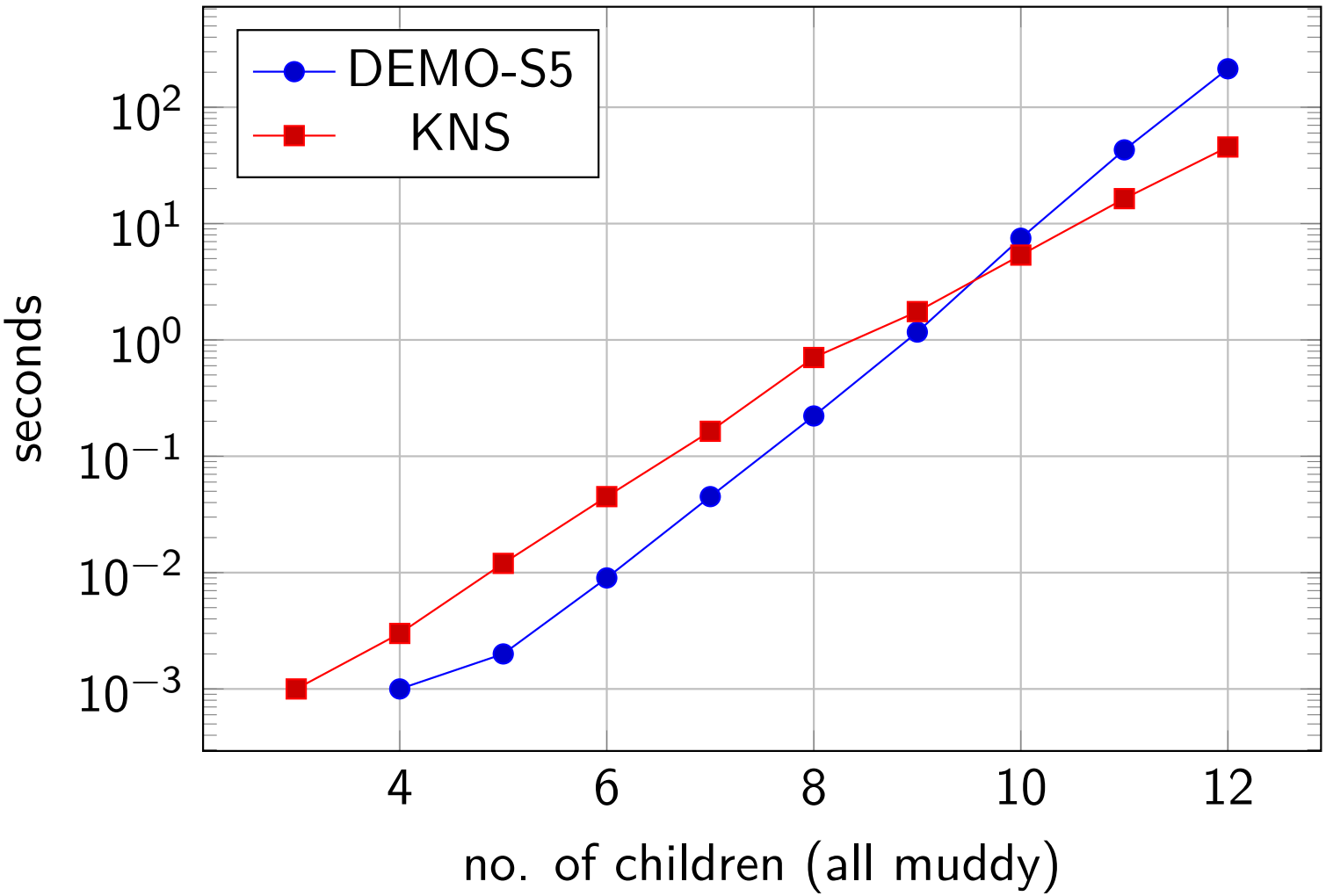
Performance

A Benchmark for Epistemic Model Checking
For n children (m of them muddy) how many announcements of ψ are needed until someone knows their own state?

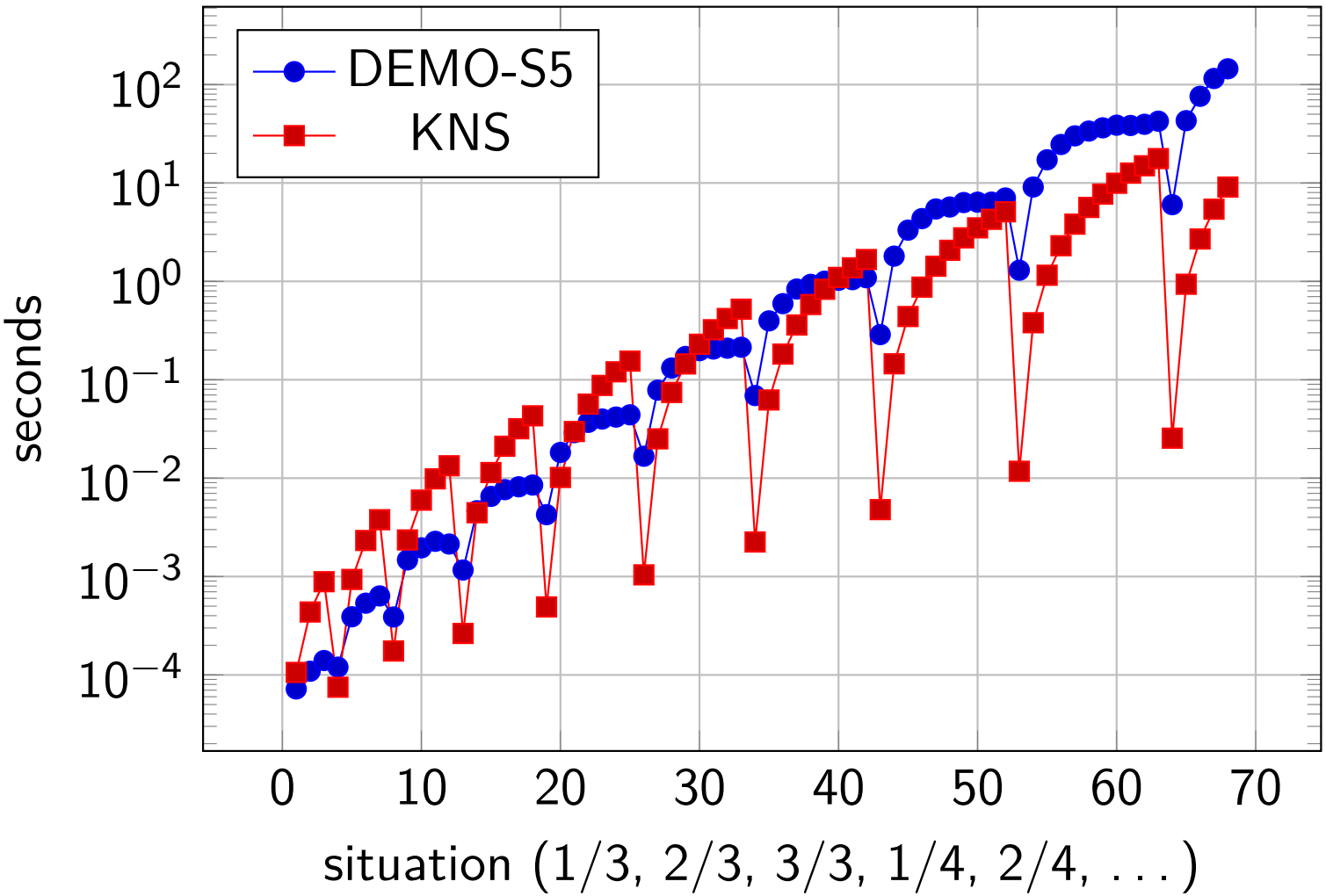
Hint: $m - 1$

We compare DEMO-S5 from [vE14] with the new BDD approach.

Performance: Overview



Performance: Details



Product Update

Group announcements

Our language also contains $[\psi]_\Delta \phi$ which says that after announcing ψ to the group Δ, ϕ holds.

On Knowledge Structures, this is true at s of

$$\mathcal{F} = (V, \theta, (O_i)_{i \in I})$$

iff $\mathcal{F}, s \vDash \psi$ implies that ϕ is true at $s \cup \{p_\psi\}$ of

$$\mathcal{F}' = (V \cup \{p_\psi\}, \theta \wedge (p_\psi \to \|\psi\|_\mathcal{F}), O'_i)_{i \in I})$$

where O'_i contains p_ψ iff $i \in \Delta$.

NB: This is *not a secret* announcement.

Now generalize this idea for any S5 product update from [BMS98].

The general case: Product Update

Similar to the product update

$$\text{Kripke Model} \otimes \text{Action Model} \to \text{Kripke Model}$$

we can define a general update mechanism

$$\text{KNS} \otimes \text{Knowledge Transformer} \to \text{KNS}$$

Definition

A *knowledge transformer* for vocabulary V is a tuple

$$\mathcal{X} = (V^+, \mu, O_1^+, \ldots, O_n^+)$$

where

1. V^+ is a set of atomic propositions such that $V \cap V^+ = \varnothing$,
2. μ is a (possibly epistemic!) formula over $V \cup V^+$ and
3. $O_i^+ \subseteq V^+$ for all agents i.

The general case: Product Update

Definition

Given $\mathcal{F} = (V, \theta, O_1, \ldots, O_n)$ and $\mathcal{X} = (V^+, \mu, O_1^+, \ldots, O_n^+)$, let

$$\mathcal{F} \otimes \mathcal{X} := (V \cup V^+, \theta \wedge ||\mu||_{\mathcal{F}}, O_1 \cup O_1^+, \ldots, O_n \cup O_n^+)$$

Theorem

For every Knowledge Transformer there is an equivalent S5 Action Model and vice versa.

Open Questions

Open Questions

- How is the performance on other puzzles and protocols? (Sum and Product, Consecutive Numbers, Russian Cards, Dining Cryptographers, . . .)
- Compare to the number triangle approach in [GS11]?
- Optimize the S5→KNS–translation by "recognizing" observational propositions before adding new ones?
- Encode non-S5 models in something similar to KNS?
- Factual change as in [BEK06]?
- Could KNS help us to find new axiomatizations or completeness proofs?

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Thank you!

Image credit: *Wacken-Besucher feiern im Schlamm*, Focus Online, 05.08.2012.